# Designing a data envelopment analysis model with a network structure and undesirable output for allocating fixed costs in bank branches

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## ABSTRACT

The fixed cost allocation in bank branches by supervisors or central headquarters has always been the concern of managers and researchers. In numerous operative applications, the units under assessment or evaluation are in a combined form, such as, mixed, in series and parallel units; and correspondingly, of an undesirable output. in this paper, we have initially presented a mixed network assessment model, with an independent input and a common intermediate, along with, an undesirable output. Subsequently, with due attention to the significance of the fixed cost allocation in relevance with mixed networks, a model for the fixed cost allocation, has been rendered, in context to the intermediate input, such that, the overall efficiency and the efficiency of the sub-units of the network do not deteriorate further, to the most probable extent after allocation, but rather, enhance to the utmost possible degree. So as to achieve an exceptional and exclusive value, a secondary goal has been taken advantage of; and this procedure, condenses the possibility of varied solutions. Likewise, CSW technique has been utilized for assessment purposes in the paper, which provides accurate results compared to the results obtained from classical models. The model proposed in this paper is ultimately implemented in an empirical example with 50 branches of one of Iran's commercial banks. the results of which, presents an example of a fair allocation, such that, the total efficiency, as well as the efficiency of the sub-units, refrain from decrease and increase as much as possible, after the allocation.

**Keywords:** Fixed Cost Allocation; Data Envelopment Analysis (DEA) Network; Mixed Network; Undesirable Output and Common set of Weights

# 1. Introduction

Data Envelopment Analysis is a technique utilized to assess the efficiency or performance of homogenous DMUs with multiple inputs and outputs. In this method, in addition to computing efficiency, other efficiency concepts, alike, ranking, density, returns to scale and fixed cost allocation are also surveyed. In each of the mentioned domains, various studies have been conducted till date and owing to the abundant application of the fixed cost allocation perception, in majority of the organizations and its role, in concern with the topic of managerial decision-making, has been brought under attentiveness in the current paper.

Today, most organizations such as banks and insurance institutions are composed of different units that operate independently under the supervision of a central organization, and in other words, integrated management is considered in those units. In these organizations, it is possible to incur costs from the central management, Like the cost of advertising and introducing new technology, which all controlled units are obliged to pay. It is clear that all units will benefit from these new services. Therefore, each of the units must to take charge a part of the new costs. While each unit has the desire to pay the minimum cost, then, the discussion of determining the fair share is raised, which is very important from an economic and managerial point of view. To answer this question, various researches have been done in recent years. Allocation of fixed cost is one of the important applications of data envelopment analysis, which is used in determining the fair share. In this technique, the fair share can be determined with the condition that the efficiency of the units under evaluation does not deteriorate after the allocation, or that all the units become efficient with this allocated cost.

The issue concerning fixed cost allocation has engrossed an immense number of scientists, from the time of its emergence. Each of them designed diverse models, on the fundaments of principles, such as, the efficiency remaining persistent, or the efficiency desisting deterioration and or an increment in efficiency after allocation. In (1999), Cook and Kress initially introduced the resource allocation model, with the objectives of sustaining efficiency and the Pareto optimization [10]. This model is beneficial where the criterions of input and output are under constant returns to scale (CRS). In (2005), Cook and Zhu, expanded the technique of Cook and Kress and rendered a more pragmatic and feasible method, though, devoid of optimization, for fixed cost allocation [11]. In (2003), Beasley, proposed an approach for fixed cost allocation, which provided an exclusive allocation, by maximizing the mean or average efficiency of units, with a set of common weights, in such a manner that, after the allocation, all the units get efficient [5]. In the year (2004), Jahanshahloo et.al demonstrated that, the Pareto optimality principle does not hold true in relevance with the Cook and Kress method; and by rendering a simple formula, they steered another approach, for the fixed cost allocation issue, without solving the linear programming problem [25]. Though, of course, the above-mentioned method only depends on the input values and does not take other factors into consideration. In accordance to which, in (2011a), Lin et.al issued a novel method for the fixed cost allocation, where, the output factors and similarly, the efficiency relativity of the DMUs, in respect to attaining the fixed cost allocation is considered alongside the aspects of input. In (2011b), Lin developed this model, by applying distinctive constraints to the Cook and Zhu model [30]. In (2013), Hosseinzadeh Lotfi et.al introduced a method for fixed cost allocation, with dual common weights. In this method, the efficiency and the objective function of the units increase after allocation [35]. Whereas, in 2009, Li et.al surveyed the correlation between the efficiency score and allocated cost, presenting a technique to seek an incomparable allotment on the fundaments of DEA [33]. In the year (2014), Du et.al suggested another method for fixed cost allocation, based on the concepts of cross-efficiency [17]. Furthermore, in (2013), Li et.al rendered the concept in relative to the degree of satisfaction, suggesting the max-min approach of the degree of satisfaction for the entire DMUs, so as to find an exceptional allocation [32]. In order to resolve the allocation issue for fixed resources, Du et.al presented an iterative method on the basis of DEA with a cross-efficiency concept [17]. In (2019), Feng et.al utilized a hybrid of Goal Programming (GP) and DEA, thus, rendering a set of common weights and determined a final allocation plan [20]. Dai et.al (2020), utilized a two-step method for fixed cost allocation for non-linear models [16]. In another research, Feng et.al (2021), in relevance with the amount of consumption of input and the amount of generated output, presented a set of common weights in favor of a fair allocation [21]. Likewise, in a combined approach, a few researchers made efforts to determine fair allocation by means of combining the game theory and DEA models. In this relative, Feng et.al (2018), introduced a new method by considering competitiveness and pliability between the DMUs, as well as the participatory game theory and the cross-efficiency method, where, this novel technique involved a unique and fair allocation plan [19]. In yet another approach, Lin et.al (2020), introduced a method for fixed cost allocation by employing a non-participatory game technique and the Nash equilibrium method [31]. In addition to the methods which have been mentioned till date, it was assumed that, cost allocation was performed on units having positive input/output and homogeneity has been accomplished, whereas, in the real-world this presumption is not essentially customary. Thereby, Ding et.al (2017) [14], as well as Lin and Chen (2020) [31], Jiasen et.al (2021), considered the fixed cost allocation in view of units with specific data [27].

As was noted, in the traditional models relevant to fixed cost allocations, in DEA, it is assumed that there is no internal structure existing in the generation or production process. Therefore, in long-established methods related to fixed cost allocations, the internal organizational processes are not brought into account; and the DMUs are considered as a black box. The only aspect taken into contemplation is the initial input in the system and the final output. In other words, in classical DEA models, this matter is not paid attention to and thus, it could be probable that, some of the sub-units could jointly utilize some of the fixed inputs of their organization. Hence, the cruciality of fixed cost allocations amidst the sub-units is an imperative facet. As a result, in the recent years, some of the researchers have performed studies in this ground, though, these studies have been limited. One of the primary studies that have been conducted in this field was by Yu et.al (2016), this was performed in relevance with two-stage units [39]. This research employed the cross-efficiency concept, in creating a fresh approach, so as to devote an appropriate mode for the allocation of fixed cost in respect to two-stage units. Correspondingly, Jiang et.al suggested a novel approach, for resource allocation, in a two-way interactive parallel system [38]. An et.al (2019), took a two-stage unit into consideration and subsequently, proposed a min-max strategy to allocate a fixed input between stages 1 and 2 by utilizing a set of common weights to assess the units [4]. In another research, Li et.al (2019), allocated a fixed input (fixed cost) amid steps 1 and 2 in such a manner that, all the sub-units are concurrently efficient after the allocation [34]. Likewise, in utilizing the participatory game theory and based on the cooperation between the DMUs, an et.al (2020), presented the fixed cost allocation among each of the units and sub-units in an inimitable way [2]. In another research An et.al (2021), introduced an approach utilizing a non-cooperative game and combining it with DEA models so as to allocate fixed costs in two-stage units, so that each unit seeks an allocation, with which the most efficiency came to hand [3]. In 2020, Chu et.al employed the Nash equilibrium and bargaining approach in game theory and combining it with DEA models proposed a new technique in order to acquire an exclusive and unique allocation in a two-stage process [12]. In each of the specified methods for fixed cost allocation, fixed cost allocation is only on two-stage units, without considering the independent inputs and outputs, nonetheless, Ding et al. (2019) developed a new hybrid method; and by manipulating the concept of degree of satisfaction and DEA, they presented a comprehensive fixed cost allocation in two-stage units, so that two-stage units, in each of the stages 1 or 2, have an independent input [15]. Correspondingly, in fixed cost allocation, this cost is only considered as an input in bearing with steps 1 and 2, though, the study conducted by Zhu et.al (2019), where initially, a fixed input allocation (fixed cost) was performed between stages 1 and 2. Subsequently a portion of the intermediate products, which was produced by stage 1 was conveyed to stage 2 and the other part exits the system [42].

In (2013), Kord Rostami et.al presented a method for production design and resource allocation in the bank, including parallel production lines, with the condition of improving efficiency after allocation. Also, Nazemi and Seiedi obtained an optimal allocation plan based on the activities of Mehr Bank branches in Khorasan Razavi province with the ABC costing method (2013). In research, Fazel et.al presented a model to evaluate the performance of banks listed on the Tehran Stock Exchange using the technique of data envelopment analysis (2014). Ahadzadeh Nemin et.al presented a cross-efficiency method based on Shannon entropy for financial evaluation of insurance companies (2018). In another study, Hosseinzadeh Lotfi et.al investigated the efficiency of 10 large petrochemical companies by modeling with the help of auxiliary variables (2018). Also, Hosseinzadeh Lotfi et.al proposed a model for evaluating the sensitivity analysis and pattern finding of the branches of one of Iran's commercial banks with relative data by using data envelopment analysis (2020). Also, in the same year, Esmailzadeh Moghri et.al evaluated the performance of 35 mutual funds as specialized financial institutions using data coverage analysis and showed that the efficiency ratio obtained from the data coverage analysis method has a significant relationship with It has the real yield of the funds (2020). After that, Hosseinzadeh Lotfi et.al investigated the effects of bank cost efficiency on the quality of financial reporting by collecting information on 12 private and commercial banks in Iran using the GMM method (2021) In the discussion of network units, Alirezaei et.al investigated the efficiency of 18 families of mutual investment funds using two-stage models of data coverage analysis (2019). In (2015), Shafiei et al proposed a two-level DEA model focusing on the multi-level structure and used linear programming for it, and then used this model to evaluate the performance of the banking chain. On the other hand, in a study, Saleh and Rostami investigated the approaches used in the banking sector and stated that two-stage models, especially in processes where input and output are shared among the processes, have more discriminating power than They have single process models and the analysis with their help is more valid (2015).

As observed in the studies conducted in relevance with fixed cost allocation, the units which are taken under consideration, are taken into account, as being, in the form of series or parallel; whereas, in several scientific applications, the units under study are a hybrid of series and parallel units (mixed) as illustrated in Fig. (2). Hence, in this paper, we first presented a fixed cost allocation model for mixed networks with independent inputs. Next, by being attentive to the importance of undesirable outputs in the assessments and an absence of an appropriate model in the research literature, for the allocation of fixed costs for units with undesirable outputs, the model presented in this paper is generalized by us, to assess units with undesirable outputs, in such a manner that, the total or overall efficiency and the efficiency of the sub-units do not deteriorate after allocation, as much as possible, but enhance, to the greatest extent conceivable. In order to achieve a select value of efficiency, a secondary objective has been employed, which diminishes the possibility of multiple solutions. Likewise, in contrary, to other DEA NETWORK MODELS, in which, each of the units under assessment has the optimum weight and is assessed optimistically, in this paper the common weight method is utilized for assessment and this renders more stringent assessment results, in comparison with outcomes achieved from classical models.

Section 2 of this paper, deals in relative to the key topic, which has been used in the proposed model, alike, DEA, an assessment of the two- stage network, this assessment model is expressed, on the basis of common weight, the fixed cost allocation technique and assessment along with an undesirable output. In section 3, the proposed model for the assessment of a distinct two-stage network is asserted and is on the fundaments of the given model, where, the allocation has been suggested in a two-stage model network model, with an additional input and an undesirable output. Finally, in section 4, the proposed model is implemented for 15 DMUs, the results of which, are assessed and investigated in the section pertaining to the conclusion.

## 2.1. Data Envelopment Analysis (DEA)

In order to assess the relative efficiency of the units under assessment, numerous approaches have been presented, which are based on estimating a function, known as the production function. The mentioned function is a function, which illustrates the maximal output for diverse input vectors. An estimation and identification of the production function is in accordance with the parametric and non-parametric methods; with the advancement of technology, parametric approaches proved to be unsuccessful in handling problems. So as to eliminate intricacies arising from parametric methods, Farrell, introduced the non-parametric method for the first time in 1957 [18]. After which, the primary idea induced by Farrell was generalized by Cooper et. al in 1978; and the CCR model was offered and this undertaking was taken to be a start, where, DEA was concerned [6].

Let us assume and consider n DMUs (DMUj $;j = 1,2,\dots n$) , which utilize M inputs $(x_{ij}; i = 1,2,\dots,M j = 1,2,\dots,n \& x_{ij} \gneq 0)$ to generate S outputs $(y_{rj}; r = 1,2,\ldots,S j = 1,2,\ldots,n \& y_{ir} \underset{\neq}{\geq} 0)$ Let us presume that, $(u_r; r = 1,2,\ldots,S)$ is the weight relative to the outputs $y_{rj}$ and $(x_i; i = 1,2,\ldots,M)$ is the weight related to the $x_{ij}$ inputs; with these presumptions, the CCR model was presented for the assessment performance of $DMU_o$ is as given below:

$$E_o = Max \sum_{r=1}^{S} u_r\, y_{ro} \quad (1)$$

$$s.t: \quad \sum_{i=1}^{M} v_i\, x_{io} = 1$$

$$\sum_{r=1}^{S} u_r\, y_{rj} - \sum_{i=1}^{M} v_i\, x_{ij} \leq 0 \quad j = 1,\ldots,n$$

$$u_r,\ v_i \geq \varepsilon \quad r = 1,\ldots,s \ \&\ i = 1,\ldots,m$$

Model (1) is known as the multiplicative form of the CCR model. If the optimum solution of model (1) equates to the value of one, the o[th] unit is 'efficient', but in the case, that, the optimum solution of the model is equivalent to zero, we describe this unit as being 'inefficient'.

## 2.2. Undesirable output in DEA

In assessing the performance or efficiency of DMUs, there is an issue, as to the manner of handling undesirable outputs that are formed alongside desirable outputs. In classical DEA models, only the desirable outputs are considered and the undesirable outputs are merely disregarded, but ignoring undesirable outputs leads to deceptive results. Thus, the DMUs should be credited for producing desirable outputs; and the unit should be reproved for producing undesirable outputs. In general, there are 5 approaches in dealing with undesirable outputs:

1) To overlook undesirable data [40] Yang and Pollitt (2009) and Halkos and Polemis (2018) [23]
2) The handling of undesirable outputs, on the same basis as desirable inputs [26] Jahanshahloo et.al (2005) and Amirteimoori et.al (2006) [1]
3) Manipulating the undesirable outputs within a mold of a non-linear model Mohd et.al (2015) [36] and Halkos and Tzeremes (2013) [24]
4) Utilizing the appropriate modifications in reference with variables in order to use this datum in the model Halkos and Papageorgiou (2014) [22] and Zanella (2014) [41]
5) Employing the issue of the principle of weak accessibility and proportionality, between the desirable and undesirable output Shephard, R.W (1970) [38].

Let us assume that in assessing n DMUs, $(DMU_j\ ;\ j = 1,2,\ldots,n)$, utilizes M inputs $(x_{ij}; i = 1,2,\ldots,M j = 1,2,\ldots,n \& x_{ij} \underset{\neq}{\geq} 0)$ to generate S desirable outputs $(y^D_{rj}; r = 1,2,\ldots,S j = 1,2,\ldots,n \& y^D_{rj} \underset{\neq}{\geq} 0)$ and T the undesirable output $(y^U_{tj}; t = 1,2,\ldots,T j = 1,2,\ldots,n \& y^U_{tj} \underset{\neq}{\geq} 0)$ is used. One of the approaches in manipulating such a circumstance of an undesirable output, is to consider this undesirable output, as an input. This model was rendered in (2006), in the configuration of model (2) by Amirteimoori et.al [1].

$$\sum_{r=1}^{S} u^o_r y_{rj} - \sum_{i=1}^{M} v^o_i x_{ij} - \sum_{t=1}^{T} w^o_t y^U_{tj} \leq 0 \quad j = 1,2,\ldots,n$$

$$u^o_r, v^o_i, w^o_t \geq \varepsilon \quad r = 1,2,\ldots,S \quad i = 1,2,\ldots,M \quad t = 1,2,\ldots,T$$

$$s.t: \quad \sum_{i=1}^{M} v^o_i x_{io} + \sum_{t=1}^{T} w^o_t y^U_{to} = 1$$

$$Max \sum_{r=1}^{S} u^o_r y^D_{ro}$$

In the mentioned model, the weight coefficients $u_r{}^o$, $w_t{}^o, v_i{}^o$; $r = 1,2,\ldots,S$, $t = 1,2,\ldots,T$, $i = 1,2,\ldots,M$ are in relevance with the desirable output $y_r{}^D$, the undesirable output $y_t{}^U$ and the $x_{ij}$ input respectively.

## 2.3. Model for common weights in DEA

A critical point which has to be paid attention to in assessing the efficiency of units by the conventional DEA models, is that, the entire DMUs select the optimal weight in order to enhance their efficiency. Thereby, the traditional DEA models usually survey the efficiency of the units benevolently. Hence, for a more precise investigation, a system comprising of models, in respect to common set of weight is given, in which, all the DMUs achieve the utmost efficiency score synchronously. Therefore, in contrary to the conventional DEA, the common weight approach, falls short of having a pliability in weights, in order to reach optimality.

One of the initial approaches for assessment in utilizing the common weight method was introduced by Cook et.al in 1990, based on resolving a goal programming (GP) problem [9]. In this method all the units under assessment are compared with a common weight and the key target of this model is to achieve the common weight $u_r, v_i$, such that, the entire units which are under assessment attain the optimal efficiency score. On the fundaments of this perception, Davoodi and Zhiani (2012), presented a model to solve the GP issue [13]; and to attain the means, in order to avoid the problem of this model, being a non-linear one, in 2013, Hosseinzadeh Lotfi et.al rendered model (3) to assess the efficiency of units under assessment and of a common weight [35].

$$\varphi = Min \sum_{j=1}^{n} \varphi_j \qquad (3)$$

$$s.t: \quad \sum_{r=1}^{S} u_r y_{rj} - \sum_{i=1}^{M} v_i x_{ij} + \varphi_j = 0 \qquad j = 1,2,\dots,n$$

$$\varphi_j \geq 0 \qquad j = 1,2,\dots,n$$

$$u_r, v_i \geq \varepsilon \qquad r = 1,2,\dots,S \qquad i = 1,2,\dots,M$$

## 2.4. Data envelopment analysis of the Multi- stage DEA

As mentioned, DEA is used as a method to assess the efficiency of analogous DMUs with multiple inputs and outputs. DMUs are put under assessment in several situations such as hospitals, universities, banks and …. In some cases, these units, function in the form of two-step process. In other words, in the first stage the inputs are utilized and outputs are hence, generated; and these outputs form the inputs of the second stage. The outputs of the first stage are known as intermediates. By utilizing these intermediates in the second stage, the final outputs of the system are produced. Fig. (1) demonstrates a two-stage DMU. In utilizing the $x_{ij}$ inputs, the $z_{dj}$ outputs are generated in the first stage. Next, in the second stage, $z_{dj}$ is considered as inputs for the production of $y_{rj}$ outputs. It should be noted that, the intermediate indexes, are regarded as outputs in the first stage, whereas, in the second stage they are viewed as inputs.

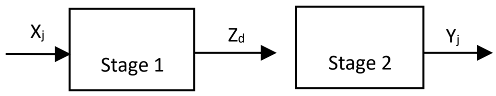



**Fig. (1) Two-stage Decision-Making Unit**

In order to assess the efficiency of two-stage units, numerous methods have been presented till date, such as, Seiford and Zhu (1999) [37], Chen and Zhu (2004) [8], Kao and Hwang (2008) [28] and Chen et.al in the year (2009) [7].

One of the most reputed models rendered in the sphere of making assessments in relevance with two-stage units, is the approach presented by Kao and Hwang (2008) [28]. In this approach, the efficiency of a two-stage network is rendered as a multiplicative attainment of efficiency of the first and second stages. This method can be expressed as follows:

$$Max \sum_{r=1}^{s} u_r y_{ro} \qquad (4)$$

$$s.t: \sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad (j = 1,\dots,n)$$

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad (j = 1,\dots,n)$$

$$w_d, u_r, v_i \geq \varepsilon, \quad (d = 1,\dots,D), (r = 1,\dots,s), (i = 1,\dots,m)$$

In this model, after computing the optimal $u_r^*, v_i^*, w_d^*$ values of the total efficiency and the efficiency of the components, they come to hand as:

$$E_o = \sum_{r=1}^{s} u_r^* y_{ro}$$
$$E_o^1 = \frac{\sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io}}$$
$$E_o^2 = \frac{\sum_{r=1}^{S} u_r^* y_{ro}}{\sum_{d=1}^{D} w_d^* z_{do}}$$

We will evidently have $E_o = E_o^1 \times E_o^2$. Mention must be made of the fact that, in model (4) there is a feasibility of attaining an alternative optimum solution; and in such a condition, secondary goals approaches can be utilized to a certain extent, so as to calculate the unique efficiency.

## 2.5. Fixed Cost Allocation

Even though DEA is a powerful technique to survey the efficiency of a set of DMUs, one of the most imperative applications of this approach, which, in the recent years, has drawn the interests of scientists in this arena, is the issue of fixed cost allocation between DMUs. Fixed cost is a cost of an organization, which has been levied on the sub-units of that organization. In other words, at times, in an organization the costs of a project prove to surpass its budget, it is essential that sub-units of that organization accept a portion of the cost, in such a manner that, the cost imposed, is divided. For example, the headquarters of a bank can invest in the producing of joint- commercial systems, for instance, the establishment and launching of ATM's etc., In such conditions, a fair approach must be pursued, so that the cost required for this project is secured from the various branches of this bank.

In the entire models, the fixed cost R, must be allocated in such a mode between the DMUs, that, primarily, each DMU considers the fixed cost $r_j$ as a new input and secondly, the efficiency of the units after allocation should not deteriorate, but alleviate to the utmost extent possible. Likewise, the total costs allocated in each DMU should be equal to the fixed cost R. Hence, constraint (5) is always supplemented to the allocation problem.

$$\sum_{j=1}^{n} r_j = R$$

Thereby, in 2009, Li et.al broached model (6) for fixed cost allocation, together with the above-mentioned conditions [33]:

$$Max \sum_{r=1}^{s} u_r y_{ro} \qquad )6($$
$$s.t: \quad \sum_{i=1}^{m} v_i x_{io} + v_{m+1} r_j = 1 \quad j = 1, \dots, n$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - v_{m+1} r_j \le 0 \quad j = 1, \dots, n$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - v_{m+1} r_j \ge E_j^* \quad j = 1, \dots, n$$
$$\sum_{j=1}^{n} r_j = R$$
$$u_r,\ v_i, v_{m+1} \ge \varepsilon\ , r_j \ge 0, r = 1, \dots, s\ \&\ i = 1, \dots, m \& j = 1, \dots, n$$

Where, in model (6) the fixed cost $r_j$ is allocated to each $\text{DMU}_j$ in such a manner, that the total costs allocated to each DMU is equivalent to the total fixed cost R, such that, after allocation the efficiency of units do not deteriorate; and to the most possible degree, enhance, from the efficiency prior to the allocation, that is, the value of $E_j^*$ improves.

## 3. Proposed model to assess the efficiency of the mixed network from the viewpoint of the undesirable output and the independent and common intermediate input

Take N decision-making units (DMUs) into consideration, in the configuration of a mixed network, as in Fig. (2).

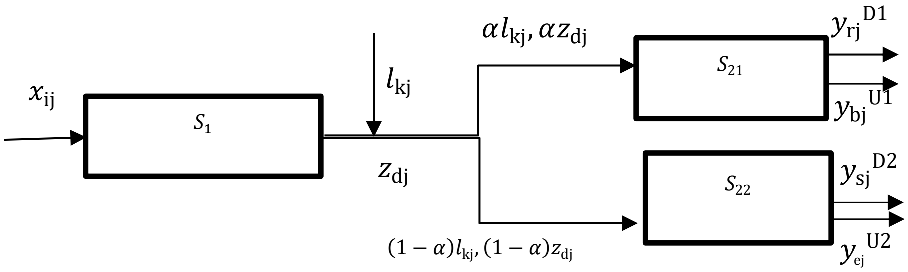



**Fig. (2) Mixed network with independent input**

In this network and in the initial stage, the sub-unit $S_1$ utilize the inputs $(x_{\mathrm{ij}}; i = 1,2,3, \dots I)$ for the production of intermediates $(z_{\mathrm{dj}}; \mathrm{d} = 1,2,3, \dots, D)$. Then, in the second stage, the intermediate produced $(z_{\mathrm{dj}}; \mathrm{d} = 1,2,3, \dots, D)$ and the additional and independent input $(l_{\mathrm{kj}}; \mathrm{k} = 1,2,3, \dots, K)$ is employed for the desirable production of $Y_j^D$ and the undesirable production of $Y_j^U$ by each of the sub-units $S_{21}$ and $S_{22}$. In other words, the sub-unit $S_{21}$ manipulates $\alpha l_{\mathrm{kj}}$ and $\alpha z_{\mathrm{dj}}$ to produce the desirable outputs $(y_{\mathrm{rj}}{}^{\mathrm{D1}}; \mathrm{r} = 1,2,3, \dots, R)$ and undesirable outputs $(y_{\mathrm{bj}}{}^{\mathrm{U1}}; \mathrm{b} = 1,2,3, \dots, B)$. Similarly, in the sub-unit $S_{22}$ , the desirable outputs $(y_{\mathrm{sj}}{}^{\mathrm{D2}}; \mathrm{s} = 1,2,3, \dots, S)$ and the undesirable output $(y_{\mathrm{ej}}{}^{\mathrm{U2}}; \mathrm{e} = 1,2,3, \dots, E)$, is produced by deploying the inputs $(1-\alpha)l_{\mathrm{kj}}$ and $(1-\alpha)z_{\mathrm{dj}}$.

To assess the efficiency of the network shown in Fig. (2), the total efficiency of the network is symbolized as $\mathrm{E_j^a}$. Correspondingly, the efficiency of the sub-unit $S_1$ is represented as $\mathrm{E_j^1}$ and the efficiency for each of the sub-units, $S_{21}$ and $S_{22}$ are denoted as $\mathrm{E_j^{21}}$ and $\mathrm{E_j^{22}}$ respectively. It should be explicated that, because the outputs $y_{bj}^{U1}$ and $y_{ej}^{U2}$ in the sub-units $S_{21}$ and $S_{22}$ are undesirable outputs and the decrement in them causes an increment in efficiency, therefore, in describing the efficiencies of $\mathrm{E_j^1}$ and $\mathrm{E_j^2}$, these outputs are observed with a negative coefficient.

With due attention to the above-mentioned subject-matter, the overall efficiency of each $\mathrm{DMU}_j$ and the efficiency of each of the mentioned sub-units is computed as given below:

$$E_j^a = \frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{\sum_{i=1}^{I} v_i x_{ij} + \sum_{k=1}^{K} t_k l_{kj} + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \quad j = 1,2,\dots,n \tag{7}$$

$$E_j^1 = \frac{\sum_{d=1}^{D} w_d z_{dj}}{\sum_{i=1}^{I} v_i x_{ij}} \quad j = 1,2,\dots,n \tag{8}$$

$$E_j^{21} = \frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1}}{\alpha(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj}) + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1}} \quad j = 1,2,\dots,n \tag{9}$$

$$E_j^{22} = \frac{\sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{(1-\alpha)(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj}) + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \quad j = 1,2,\dots,n \tag{10}$$

As a result, to compute the efficiency of each of the units and to seek the common and optimal set of weights, under which, all the units acquire the optimum efficiency score, the following multi-objective model is proposed, so as to maximize the total efficiency of each unit:

$$E^a = Max \left\{ \frac{\sum_{r=1}^{R} u_r^1 y_{r1}^{D1} + \sum_{s=1}^{S} u_s^2 y_{s1}^{D2}}{\sum_{i=1}^{I} v_i x_{i1} + \sum_{k=1}^{K} t_k l_{k1} + \sum_{b=1}^{B} h_b^1 y_{b1}^{U1} + \sum_{e=1}^{E} h_e^2 y_{e1}^{U2}}, \dots, \frac{\sum_{r=1}^{R} u_r^1 y_{rn}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sn}^{D2}}{\sum_{i=1}^{I} v_i x_{in} + \sum_{k=1}^{K} t_k l_{kn} + \sum_{b=1}^{B} h_b^1 y_{bn}^{U1} + \sum_{e=1}^{E} h_e^2 y_{en}^{U2}} \right\}$$

$$s.t: \quad \frac{\sum_{d=1}^{D} w_d z_{dj}}{\sum_{i=1}^{I} v_i x_{ij}} \le 1 \qquad j = 1,2,\dots,n \tag{11}$$

$$\frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1}}{\alpha(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj}) + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1}} \le 1 \qquad j = 1,2,\dots,n$$

$$\frac{\sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{(1-\alpha)(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj}) + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \le 1 \qquad j = 1,2,\dots,n$$

$$\frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{\sum_{i=1}^{I} v_i x_{ij} + \sum_{k=1}^{K} t_k l_{kj} + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \le 1 \qquad j = 1,2,\dots,n$$

$$u_r^1, u_s^2, h_b^1, h_e^2, w_d, v_i, t_k, \alpha, \ge \varepsilon \qquad \forall r, \forall i, \forall s, \forall d, \forall e, \forall b, \forall k$$

As can be noted, model (11) is a multi-objective problem. In order to solve this multi-objective problem, the proposed method, suggests the utilization of the Goal Programming (GP) approach. In this technique, for each of the objective functions, a goal level, $A_j$ is taken under consideration; and so that, each objective function achieves its target level, deviating variables, $\varphi_j^- and\ \varphi_j^+$ are defined. Next, for the purpose of minimizing the denoted deviating variables, efforts are made to get each objective function nearer to the target$A_j$. Thus, this model (12) is rendered as given hereunder:

$$\varphi = Min \sum_{j=1}^{n} (\varphi_j^- + \varphi_j^+) \qquad (12)$$

$$s.t:$$

$$\frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{\sum_{i=1}^{I} v_i x_{ij} + \sum_{k=1}^{K} t_k l_{kj} + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} + \varphi_j^- - \varphi_j^+ = A_j \quad j = 1,2,\dots,n \qquad (12\text{-}1)$$

$$\frac{\sum_{d=1}^{D} w_d z_{dj}}{\sum_{i=1}^{I} v_i x_{ij}} \le 1 \qquad j = 1,2,\dots,n \qquad (12\text{-}2)$$

$$\frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1}}{\alpha(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj}) + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1}} \le 1 \quad j = 1,2,\dots,n \qquad (12\text{-}3)$$

$$\frac{\sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{(1-\alpha)(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj}) + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \le 1 \quad j = 1,2,\dots,n \qquad (12\text{-}4)$$

$$\frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{\sum_{i=1}^{I} v_i x_{ij} + \sum_{k=1}^{K} t_k l_{kj} + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \le 1 \quad j = 1,2,\dots,n \qquad (12\text{-}5)$$

$$u_r^1, u_s^2, h_b^1, h_e^2, w_d, v_i, t_k, \alpha, \varphi_j^-, \varphi_j^+ \ge \varepsilon \qquad \forall r, \forall i, \forall s, \forall d, \forall e, \forall b, \forall k \qquad (12\text{-}6)$$

With due attention to the fact that, the objective functions of model (12) are of efficiency types, therefore, the goal level of each of the constraints (12-1) equates to 1. So, $A_j = 1$ can be placed in model (12). Whereas, by being attentive towards a set of constraints (12-5), the objective function must be increased, in order to achieve the signified target. Hence, to gain access to this goal, on the conceptual basis of the paper of Hosseinzadeh Lotfi et.al (2013) [35]; the constraints (12-1) can be modified in such a manner that the fraction increments. (Increasing the numerator by adding $\varphi_j^+$ and decreasing the denominator by subtracting $\varphi_j^-$, augments the fraction as much as possible). So, in model (12), the constraint (1-12) can be converted to (13). Hence, the set of restraints (12-5) is Redundant for this model.

$$\frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2} + \varphi_j^+}{\sum_{i=1}^{I} v_i x_{ij} + \sum_{k=1}^{K} t_k l_{kj} + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{e=1}^{E}} = 1 \qquad j = 1,2,\dots,n \qquad )13($$

As can be observed, model (12) is non-linear. As a result, modifying variable (14) for linearization, is defined as hereunder:

$$\begin{cases} \alpha w_d = f_d & d = 1,2,\dots,D \\ \alpha t_k = g_k & k = 1,2,\dots,K \\ \varphi_j^- + \varphi_j^+ = \varphi_j & j = 1,2,\dots,n \end{cases} \qquad (14)$$

In this manner, the linear model (15) can be easily attained to assess the efficiency of the mentioned network.

$$\varphi = Min \sum_{j=1}^{n} \varphi_j \qquad (15)$$

$s.t:$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} - \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} - \sum_{i=1}^{I} v_i x_{ij} - \sum_{k=1}^{K} t_k l_{kj} + \varphi_j = 0 \qquad j = 1,2,\dots,n$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} - \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} - \sum_{d=1}^{D} f_d z_{dj} - \sum_{k=1}^{K} g_k l_{kj} \le 0 \qquad j = 1,2,\dots,n$$

$$\sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} - \sum_{d=1}^{D} w_d z_{dj} + \sum_{d=1}^{D} f_d z_{dj} - \sum_{k=1}^{K} t_k l_{kj} + \sum_{k=1}^{K} g_k l_{kj} \le 0 \qquad j = 1,2,\dots,n$$

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{I} v_i x_{ij} \le 0 \qquad j = 1,2,\dots,n$$

$$u_r^1, u_s^2, h_b^1, h_e^2, w_d, v_i, t_k, g_k, f_d \ge \varepsilon, \varphi_j \ge 0 \quad \forall r, \forall i, \forall s, \forall d, \forall e, \forall b, \forall k$$

**Definition 1:** In the O th unit )DMUo(, the total efficiency is equivalent to 1, If and only if in model (15), $\varphi_O{}^* = 0$

**Theorem 1:** Model (15) is always feasible.

**Proof:** It can be demonstrated easily. There is one conceivable solution for model (15)

$$u_r^1 = \begin{cases} \frac{1}{y_{1j}^{D1}} r = 1 \\ 0 o.w \end{cases} ; r = 1,\dots,R \,\&\, u_s^2 = \begin{cases} \frac{1}{y_{1j}^{D2}} s = 1 \\ 0 o.w \end{cases} ; s = 1,\dots,S \,\&\, h_b^1 = \begin{cases} \frac{1}{y_{1j}^{U1}} b = 1 \\ 0 o.w \end{cases} ; b = 1,\dots,B$$

$$h_e^2 = \begin{cases} \frac{1}{y_{1j}^{2}} e = 1 \\ 0 o.w \end{cases} ; e = 1,\dots,E \,\&\, f_d = \begin{cases} \frac{1}{z_{1j}} d = 1 \\ 0 o.w \end{cases} ; d = 1,\dots,D \,\&\, w_d = \begin{cases} \frac{1}{w_{1j}} d = 1 \\ 0 o.w \end{cases} ; d = 1,\dots,D$$

$$g_k = \begin{cases} \frac{1}{l_{1j}} k = 1 \\ 0 o.w \end{cases} ; k = 1,\dots,K \,\&\, t_k = \begin{cases} \frac{1}{l_{1j}} k = 1 \\ 0 o.w \end{cases} ; k = 1,\dots,K \,\&\, v_i = \begin{cases} \frac{1}{x_{1j}} i = 1 \\ 0 o.w \end{cases} ; i = 1,\dots,I$$

$$\varphi_k = \begin{cases} 2 k = j \\ 0 o.w \end{cases}$$

Hence, this model is always feasible.

**Theorem 2:** In model (15), $\varphi^* = 0$; if and only if the total efficiency of the entire units under assessment equates to 1.

**Proof:**

$$\varphi^* = 0 \leftrightarrow \sum_{j=1}^{n} \varphi^*{}_j = 0 \xleftrightarrow{\varphi^*{}_j \ge 0} \forall j, \varphi^*{}_j = 0$$

If, $\varphi^*{}_j = 0$, if in accordance with definition (1) it can be said that it is efficient and vice-versa.

If there are multiple optimal solutions in model (15), diverse methods of secondary goals can be manipulated to gain the total efficiency and the efficiency of the subunits:

1) For each DMU a problem is solved. In other words, in this approach, the optimum value comes to hand for each DMU, as to the best solutions of model (15) and in this approach, it can be confidently stated that the secondary problem has an inimitable solution.
2) The secondary target, in integrating the target of the entire DMUs into one problem and resolving it, in which case, there is no

necessity for the presence of an exclusive solution. In this paper the first method is utilized to bring about a uniqueness for the optimal solution.

3)

### 3.1. A secondary goal to make the optimum solution unique

Short of perturbing and creating a void in the generalities of the subject matter, we will assume that, model (15), has an optimum and multiple solution, to attain the optimal unique weights and to seek the overall efficiency value, as well as the efficiency of the components; and the value of the efficiency of each unit, in the set of varied solutions of that unit, is maximized. For example, the total efficiency value of unit number L, in regards the multiple optimal solutions of model (15) is increased as much as achievable with the following model.

$$E_L^{a*} = Max\frac{\sum_{r=1}^{R} u_r^1 y_{rL}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sL}^{D2}}{\sum_{i=1}^{I} v_i x_{iL} + \sum_{k=1}^{K} t_k l_{k1} + \sum_{b=1}^{B} h_b^1 y_{bL}^{U1} + \sum_{e=1}^{E} h_e^2 y_{eL}^{U2}} \quad (16)$$

$$s.t: \varphi^* = \sum_{j=1}^{n} \varphi_j \quad (16\text{-}1)$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} - \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} - \sum_{i=1}^{I} v_i x_{ij} - \sum_{k=1}^{K} t_k l_{kj} + \varphi_j = 0 \quad j = 1,2,\dots,n \quad (16\text{-}2)$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} - \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} - \sum_{d=1}^{D} f_d z_{dj} - \sum_{k=1}^{K} g_k l_{kj} \le 0 \quad j = 1,2,\dots,n \quad (16\text{-}3)$$

$$\sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} - \sum_{d=1}^{D} w_d z_{dj} + \sum_{d=1}^{D} f_d z_{dj} - \sum_{k=1}^{K} t_k l_{kj} + \sum_{k=1}^{K} g_k l_{kj} \le 0 \quad j = 1,2,\dots,n \quad (16\text{-}4)$$

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{I} v_i x_{ij} \le 0 \quad j = 1,2,\dots,n \quad (16\text{-}5)$$

$$u_r^1, u_s^2, h_b^1, h_e^2, w_d, v_i, t_k, g_k, f_d, \varphi_j \ge \varepsilon \quad \forall r, \forall i, \forall s, \forall d, \forall e, \forall b, \forall k \quad (16\text{-}6)$$

If in the set of constraints (16-2) the value is $\varphi_L^* = 0$, this signifies that the value of the L[th] unit in the different solution sets of this unit, increases to the utmost possible degree. Hence, in model (16), the objective function can be modified, by targeting for a maximum decrease in $\varphi_L$, on the basis of which, model (17) is manipulated as a secondary goal, in order to find an exclusive solution to the utmost extent in model (15) and utilized for each DMU$_L$.

$$E_L^{a*} = min\, \varphi_L \quad )17($$

$$s.t: \varphi^* = \sum_{j=1}^{n} \varphi_j$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} - \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} - \sum_{i=1}^{I} v_i x_{ij} - \sum_{k=1}^{K} t_k l_{kj} + \varphi_j = 0 \quad j = 1,2,\dots,n$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} - \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} - \sum_{d=1}^{D} f_d z_{dj} - \sum_{k=1}^{K} g_k l_{kj} \le 0 \quad j = 1,2,\dots,n$$

$$\sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} - \sum_{d=1}^{D} w_d z_{dj} + \sum_{d=1}^{D} f_d z_{dj} - \sum_{k=1}^{K} t_k l_{kj} + \sum_{k=1}^{K} g_k l_{kj} \le 0 \quad j = 1,2,\dots,n$$

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{I} v_i x_{ij} \le 0 \quad j = 1,2,\dots,n$$

$$u_r^1, u_s^2, h_b^1, h_e^2, w_d, v_i, t_k, g_k, f_d \ge \varepsilon, \varphi_j \ge 0 \quad \forall r, \forall i, \forall s, \forall d, \forall e, \forall b, \forall k$$

### 3.2. A proposed model to assess the efficiency of fixed cost allocation in a mixed network with undesirable output and independent input and a common intermediate

In this section we presume that, cost $c_j$, which is considered as a new input is allotted to the independent intermediate input $l_j$ in such a manner, that, the overall efficiency and the efficiency of the entire sub-units, after the allocation, do not deteriorate as much as is conceivable, then before, but alleviate or improve as much as much as possible. In other words, the new $L_j$ is described as $L_j = (l_{1j}, l_{2j}, \dots, l_{kj}, c_j)$. Likewise, the total allocation cost attained for $c_j$, should be equivalent to the fixed cost of C, $\sum_{j=1}^{n} c_j = C$ ; if, $t_{k+1}$ is the weight coefficient, in relative to the variable $c_j$, the total efficiency of each unit and the efficiency of each of the mentioned sub-units are computed as follows:

$$E_j^a = \frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{\sum_{i=1}^{I} v_i x_{ij} + \sum_{k=1}^{K} t_k l_{kj} + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} + t_{k+1} c_j} \quad j = 1,2,\dots,n \qquad (18)$$

$$E_j^1 = \frac{\sum_{d=1}^{D} w_d z_{dj}}{\sum_{i=1}^{I} v_i x_{ij}} \qquad j = 1,2,\dots,n \qquad (19)$$

$$E_j^{21} = \frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1}}{\alpha(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj} + t_{K+1} c_j) + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1}} \qquad j = 1,2,\dots,n \qquad (20)$$

$$E_j^{22} = \frac{\sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{(1-\alpha)(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj} + t_{K+1} c_j) + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \qquad j = 1,2,\dots,n \qquad (21)$$

Each DMU$_j$ has an inclination towards maximizing each of the above-mentioned efficiencies. Thereby, the multi-objective model below, targets to maximize the overall efficiency of each unit bearing a common weight, on condition that, the total efficiency of all the sub-units, improve after allocation till a feasible degree; and correspondingly, the overall allocated cost of $c_j$ equates to the fixed cost of C. This is proposed in the following configuration:

$$max \left\{ \frac{\sum_{r=1}^{R} u_r^1 y_{r1}^{D1} + \sum_{s=1}^{S} u_s^2 y_{s1}^{D2}}{\sum_{i=1}^{I} v_i x_{i1} + \sum_{k=1}^{K} t_k l_{k1} + \sum_{b=1}^{B} h_b^1 y_{b1}^{U1} + \sum_{e=1}^{E} h_e^2 y_{e1}^{U2} + t_{k+1} c_1}, \dots, \frac{\sum_{r=1}^{R} u_r^1 y_{rn}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sn}^{D2}}{\sum_{i=1}^{I} v_i x_{in} + \sum_{k=1}^{K} t_k l_{kn} + \sum_{b=1}^{B} h_b^1 y_{bn}^{U1} + \sum_{e=1}^{E} h_e^2 y_{en}^{U2} + t_{k+1} c_n} \right\}$$

(22)

$$s.t: \quad E_j^{1*} \le \frac{\sum_{d=1}^{D} w_d z_{dj}}{\sum_{i=1}^{I} v_i x_{ij}} \le 1 \qquad j = 1,2,\dots,n$$

$$E_j^{21*} \le \frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1}}{\alpha(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj} + t_{K+1} c_j) + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1}} \le 1 \qquad j = 1,2,\dots,n$$

$$E_j^{22*} \le \frac{\sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{(1-\alpha)(\sum_{d=1}^{D} w_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj} + t_{K+1} c_j) + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \le 1 \qquad j = 1,2,\dots,n$$

$$E_j^{a*} \le \frac{\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2}}{\sum_{i=1}^{I} v_i x_{ij} + \sum_{k=1}^{K} t_k l_{kj} + t_{K+1} c_j + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}} \le 1 \qquad j = 1,2,\dots,n$$

$$\sum_{j=1}^{n} c_j = C$$

$$u_r^1, u_s^2, h_b^1, h_e^2, w_d, v_i, t_k, t_{K+1}, \alpha, c_j \ge \varepsilon \qquad \forall r, \forall i, \forall s, \forall d, \forall e, \forall b, \forall k, \ \forall j$$

Such that, $E_j^{22*}, E_j^{21*}, E_j^{1*}, E_j^{a*}$ are respectively, the optimum and total efficiency values and the efficiency of the stages $S_1$, $S_{21}$ and $S_{22}$ after allocation.

It is obvious that model (22) is a multi-objective problem and hence to solve the mentioned problem, we utilize the GP approach. In a similar manner for the process stated in model (12) in section 3; and presenting the deviating variables $aa_j, a1_j, a21_j, a22_j$; in order to improve the total efficiency and the efficiency of components to a feasible extent, besides modifying variable(s) (23), we shall put across model (24) for the fixed cost allocation:

$$\begin{cases} \alpha w_d = f_d & d = 1,2,\dots,D \\ \alpha t_k = g_k & k = 1,2,\dots,K \\ t_{k+1} c_j = \overline{c_j} & j = 1,2,\dots,n \\ \alpha \overline{c_j} = \beta_j & j = 1,2,\dots,n \end{cases} \qquad (23)$$

$$\omega = min \sum_{j=1}^{n} (a_j^1 + a_j^2 + a_j^3 + a_j^4)$$

(24)

$s.t$:

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{I} v_i x_{ij} \le 0 \qquad j = 1,2,\dots,n$$

$$\sum_{d=1}^{D} w_d z_{dj} - E_j^{1*} \sum_{i=1}^{I} v_i x_{ij} \ge a_j^1 \qquad j = 1,2,\dots,n$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - \sum_{i=1}^{I} v_i x_{ij} - \sum_{k=1}^{K} t_k l_{kj} - \overline{c_j} - \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} - \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} \leq 0 \qquad j = 1,2,\dots,n$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} + \sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - E_j^{a*}\left(\sum_{i=1}^{I} v_i x_{ij} + \sum_{k=1}^{K} t_k l_{kj} + \overline{c}_j + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1}\right) \geq a_j^4 \qquad j = 1,2,\dots,n$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} - \sum_{d=1}^{D} f_d z_{dj} - \sum_{k=1}^{K} g_k l_{kj} - \beta_j - \sum_{b=1}^{B} h_b^1 y_{bj}^{U1} \leq 0 \qquad j = 1,2,\dots,n$$

$$\sum_{r=1}^{R} u_r^1 y_{rj}^{D1} - E_j^{21*}\left(\sum_{d=1}^{D} f_d z_{dj} + \sum_{k=1}^{K} g_k l_{kj} + \beta_j + \sum_{b=1}^{B} h_b^1 y_{bj}^{U1}\right) \geq \varphi \qquad j = 1,2,\dots,n$$

$$\sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - \sum_{d=1}^{D} w_d z_{dj} + \sum_{d=1}^{D} f_d z_{dj} - \sum_{k=1}^{K} t_k l_{kj} + \sum_{k=1}^{K} g_k l_{kj} - \overline{c_j} + \beta_j - \sum_{e=1}^{E} h_e^2 y_{ej}^{U2} \leq a_j^2 \qquad j = 1,2,\dots,n$$

$$\sum_{s=1}^{S} u_s^2 y_{sj}^{D2} - E_j^{22*}\left(\sum_{d=1}^{D} w_d z_{dj} - \sum_{d=1}^{D} f_d z_{dj} + \sum_{k=1}^{K} t_k l_{kj} - \sum_{k=1}^{K} g_k l_{kj} + \overline{c_j} - \beta_j + \sum_{e=1}^{E} h_e^2 y_{ej}^{U2}\right) \geq a_j^3 \qquad j = 1,2,\dots,n$$

$$\sum_{j=1}^{n} \bar{c}_j = C * t_{K+1}$$

$$u_r^1, u_s^2, h_b^1, h_e^2, w_d, v_i, t_k, t_{K+1}, g_k, f_d, \beta_j \geq \varepsilon \ , a_j^1, a_j^2, a_j^3, a_j^4 \geq 0 \qquad \forall r, \forall i, \forall s, \forall d, \forall e, \forall b, \forall k,\ \forall j$$

**Definition 2:** In the O th **unit (**DMUo), the total efficiency after allocation does not get worse, when compared to the efficiency prior the allocation, if and only if, in model (24) $a_o^{4*} = 0$.

**Definition 3:** In the O th **unit (**DMUo), the efficiency of the sub-unit $S_1$ does not decline in comparison to the efficiency before the allocation, if and only if, $a_O^{1*} = 0$ in model (24).

**Definition 4:** In the O th **unit (**DMUo), the efficiency of the sub-unit $S_{21}$ does not decrease in comparison to the efficiency before the allocation, if and only if, $a_O^{2*} = 0$ in model (24).

**Definition 5:** In the O th **unit (**DMUo), the efficiency of the sub-unit $S_{22}$ does not deteriorate compared to the efficiency preceding to the allocation, if and only if, $a_O^{3*} = 0$ in model (24).

**Theorem 3:** In model (24) $\omega^* = 0$, if and only if, the overall efficiency as well as the efficiency of the components do not decline in comparison to the former.

**Proof:**

$$\omega^* = 0 \leftrightarrow \sum_{j=1}^{n} (aa^*{}_j + a1^*{}_j + a21^*{}_j + a22^*{}_j) = 0 \xleftrightarrow{a^{1*}{}_j \geq 0, a^{2*}{}_j \geq 0 a^{3*}{}_j \geq 0, a^{4*}{}_j \geq 0} \forall j; a^{1*}{}_j = 0, a^{2*}{}_j = 0, a^{3*}{}_j = 0, a^{4*}{}_j = 0$$

If, $\forall j, , a^{1*}{}_j = 0, a^{2*}{}_j = 0, a^{3*}{}_j = 0, a^{4*}{}_j = 0$ , in accordance with definitions (2 to 5), it can be expressed that the total efficiency and the efficiency of the components for each $DMU_j; j = 1, \dots, n$ after allocation does not worsen and vice-versa.

**Theorem 4:** Model (24) is always feasible.

**Proof:**

It can be straightforwardly demonstrated that:

$$u_r^1 = \begin{cases} \frac{3p}{y_{1j}^{D1}} r = 1 \\ 0 o.w \end{cases} ; r = 1, \dots, R \& u_s^2 = \begin{cases} \frac{3p}{y_{1j}^{D2}} s = 1 \\ 0 o.w \end{cases} ; s = 1, \dots, S \ \& h_b^1 = \begin{cases} \frac{2p}{y_{1j}^{U1}} b = 1 \\ 0 o.w \end{cases} ; b = 1, \dots, B$$

$$h_e^2 = \begin{cases} \frac{2p}{y_{1j}^2} e = 1 \\ 0 o.w \end{cases} ; e = 1, \dots, E \& f_d = \begin{cases} \frac{p}{3z_{1j}} d = 1 \\ 0 o.w \end{cases} ; d = 1, \dots, D \ \& w_d = \begin{cases} \frac{3p}{w_{1j}} d = 1 \\ 0 o.w \end{cases} ; d = 1, \dots, D$$

$$g_k = \begin{cases} \frac{p}{3l_{1j}} & k = 1 \\ 0 & o.w \end{cases} ; k = 1,\dots,K \quad \& t_k = \begin{cases} \frac{p}{l_{1j}} & k = 1 \\ 0 & o.w \end{cases} ; k = 1,\dots,K \quad \& v_i = \begin{cases} \frac{3p}{x_{1j}} & i = 1 \\ 0 & o.w \end{cases} ; i = 1,\dots,I$$

$$a_j^1 = 0 \ \& a_j^2 = 0 \& a_j^3 = 0 \& a_j^4 = 0 \& \ \overline{C}_n = \begin{cases} p & n = j \\ 0 & o.w \end{cases} \&; n = 1,\dots,N \& \beta_n = \begin{cases} \frac{p}{3} & n = j \\ 0 & o.w \end{cases} ; n = 1,\dots,N$$

it is a possible solution for model (24); hence, this model is always feasible.

In this section moreover, if there are alternative optimum solutions in model (24), the secondary goal technique can be used to attain the total efficiency and the efficiency of the subunits.

## 4. Fixed cost allocation in bank branches

Financial institutions are accounted for as being the most vital economic institutions of a country, and the growth and opulence of the country's economy rests on their augmentation and development. If the financial markets lack development, the above-mentioned institutions will also not have the necessary efficiency. It seems that, the only manner to develop the economy of a country, which will lead to an elevation in the level of welfare of a nation, is an increment of productivity in the economic institutions of that nation state. Incredible transformations in growth and economic development on a short-term basis, in some countries, such as, Japan, Germany, China and a number of East Asian countries have been the consequences of amplified productivity and the optimal, efficient and effectual usage of their countries' physical and human resources. Banks are reputedly known as significant institutions in the financial markets, as it seems that the efficiency and performance of banks in every country is of deserving importance and ranking. Thence, in this section, an assessment of the efficiency in relative to the branches of one of the Iranian commercial banks have been brought into view. Such that, every j branch, as is illustrated in Fig. (1), is in the configuration of a mixed network; so that, in the first stage it has 1 initial **(X)** input and 2 intermediate outputs ($Z_1$, $Z_2$), which are totally consumed in the second stage. The second stage comprises of two sub-units, each of which, is for the purpose of generating the final output, consume a segment of the intermediate output generated by the first stage and the intermediate independent input **(L)**. In other words, in the second stage, the initial sub-units consume the inputs $\alpha$L , $\alpha Z_1, \alpha Z_2$ and produce two desirable outputs $Y_1{}^{D1}, Y_2{}^{D1}$ and $Y^{U1}$, which is the undesirable output produced. In a similar way, the second sub-units, by consuming the $(1-\alpha)$L , $(1-\alpha)Z_1, (1-\alpha)Z_2$ inputs, generate the desirable and undesirable outputs $Y^{D2}$ and $Y^{U2}$ respectively.

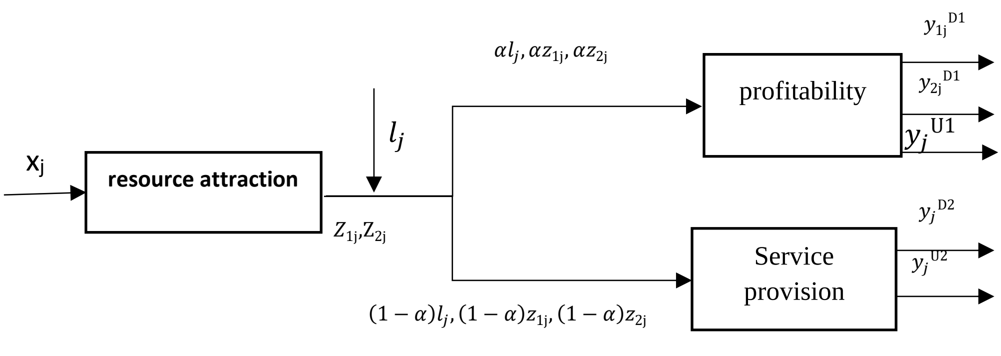



**Fig. (3) Two-stage mixed network in relevance with the j^th bank branch**

For each branch, j, which represents a DMU, two inputs, $x_j$ and $l_j$ designate the privilege of staff, in the configurations of the input $x_j$ in the first stage; and as a $l_j$ input in the second stage, which, enters the network, in the form of an intermediate. It should be elucidated that, the privilege of staff, is an index attained from the number of personnel, their employment record, the educational qualifications of the personnel and their level of training. This is computed from their weighted sum, on the founding of weights that illustrate the importance of these sub-indexes. Their prominence in order of merit, is 0.35, 0.27, 0.15 and 0.23 respectively. Correspondingly, the five final outputs of the system comprise of two desirable outputs $y_{1j}^{D1}$ and $y_{2j}^{D1}$ signifying the facilities or (convenience) granted and earnings attained and one undesirable output $y_j^{U1}$ known as deferred payments; outputs related to the sub-unit $S_{21}$ and a desirable output $y_j^{D2}$ designating the fees charged and one undesirable output $y_j^{U2}$, which represents the interest paid; and is defined as the output of the sub-unit $S_{22}$. Each of these outputs have been described in segregation, in accordance with the Iranian banking system.

- ✓ The granted facilities or convenience is an index to illustrate the total amount of loans, which a bank pays its clientele within a time-period of one month and its unit is millions of Rials
- ✓ The interest earned is a percentage of the loan or facility, which the bank receives from its clientele in lieu for the payment of that facility. The various loans have varied profits. In total, the general profit or interests receives for the entire preceding loans, is known as the interest received and its unit of measurement is millions of Rials
- ✓ Deferred claims are the amounts of overdue or a lack of payment by the clientele for the installments of loans and its unit of measurement is millions of Rials. We are aware of the fact that, one of the banking activities in order to earn revenue, is the granting of loans to clientele. On receiving the loan(s), it is obligatory, that the clients repay the loan in monthly installments to the bank, but unfortunately, these loans get overdue and are recompensed with delay, and or the installments are not reimbursed at all. An increment in these arrears will cause a decline in the efficiency of the branch. Hence, this index is contemplated upon, as an undesirable output
- ✓ The fee received, all banks render several types of services to their clientele, such as cash transfer, rendering guarantees and … In lieu of these services to their clients, the banks receive a fee from their customers, also known as bank charges. An aggregation of fees or bank charges or fee, which a bank receives per month for all the services granted, is called the monthly fee and its measurement unit is millions of Rials
- ✓ Profits or the Interests paid, banks pay interest for the deposits of their clients. For ‘Gharz-al-Hassaneh’ deposits no interest is paid. Though, for long- and short-term deposits, interest is necessitated. The total of all the interests that the bank renders per month to the entire long- and short-term deposits, is called interest paid and its measurement unit is millions of Rials.

Moreover, the sum of deposits and other resources, which are given and presented as intermediate products $z_{1j}$ and $z_{2j}$ and are produced in the first stage and as can be observed in Fig. (3), $\alpha z_{1j}$ and $\alpha z_{2j}$ in stage $S_{21}$ and likewise, $(1-\alpha)z_{1j}$ and $(1-\alpha)z_{2j}$ are consumed entirely in stage $S_{22}$.

The sum of deposits implies to the cumulative or aggregate of deposits of clientele, in the segments of savings and current savings pertaining to Gharz-al - Hassaneh, as well as the long- term and short-term savings of clients in a bank. In actuality, the balance of these deposits is in a month, that, is the mean or average of the total number of days in a month and its unit is millions of Rials. Correspondingly, other resources are the number of deposits of the entire customers, with exception of the sum of deposits (4 deposits), other resources are being presented as sample resources, which the government has in banks. The values in relevance with each one of the indexes on the fundaments of the documents accessible in the banking system, for each of the branches, are attested in Table (1).

**Table (1) Input and output values consumed in each stage**

| The number of unit | Primary personnel | Middle personnel | Deposits | Other resorces | Granted facilites | Interest earned | Deferred claims | Fee recived | Profits paid |
|---|---|---|---|---|---|---|---|---|---|
| $DMU_j$ | $x_j$ | $l_j$ | $z_{1j}$ | $z_{2j}$ | $y_{1j}^{D1}$ | $y_{2j}^{D1}$ | $y_j^{U1}$ | $y_j^{D2}$ | $y_j^{U2}$ |
| 1 | 8/15 | 8/15 | 44981778745 | 238309198 | 106520333464 | 38535959 | 1373678927 | 59265020 | 570344468 |
| 2 | 2/72 | 2/72 | 18662637612 | 0 | 42283737772 | 603927289 | 1186694577 | 2851150 | 170703908 |
| 3 | 1/63 | 1/63 | 15536642810 | 67950000 | 73802496456 | 164426685 | 1895166871 | 25798743 | 151328572 |
| 4 | 5/27 | 5/27 | 49472817885 | 258667100 | 171597270712 | 360807532 | 1705000 | 40167368 | 567732758 |
| 5 | 2/86 | 2/86 | 11610302656 | 38200000 | 31221989205 | 351892976 | 506735784 | 11573507 | 157863965 |
| 6 | 2/10 | 2/10 | 15197849180 | 0 | 21999695664 | 38720593 | 164674589 | 1433588 | 124775991 |
| 7 | 16/99 | 16/99 | 91723395046 | 175385400 | 220469463681 | 906452730 | 2086168200 | 72724247 | 1079358382 |
| 8 | 6/58 | 6/58 | 56454519789 | 191869700 | 101333689373 | 1458250548 | 121796222 | 37738410 | 747456958 |
| 9 | 3/02 | 3/02 | 23518104429 | 166545000 | 31746964459 | 87275923 | 164783100 | 26690662 | 311866711 |
| 10 | 9/32 | 9/32 | 72423260668 | 939864659 | 158690471008 | 947080100 | 1182161972 | 47258172 | 1031120275 |
| 11 | 2/27 | 2/27 | 15253588659 | 4290000 | 44842531204 | 223776129 | 820113418 | 29136145 | 209183080 |
| 12 | 4/19 | 4/19 | 41404997069 | 0 | 66659454572 | 413047000 | 381179730 | 15120618 | 386349703 |
| 13 | 5/50 | 5/50 | 30269717088 | 11934800 | 50609705055 | 288274690 | 213817809 | 52000292 | 378307741 |
| 14 | 8/16 | 8/16 | 70768586794 | 140269880 | 83225550325 | 256068115 | 6749810197 | 84525013 | 730377583 |
| 15 | 2/13 | 2/13 | 20226184747 | 51500000 | 34481739099 | 166913188 | 58233105 | 13546827 | 213340711 |
| 16 | 2/05 | 2/05 | 21861219886 | 32933234 | 29115005183 | 190594418 | 10000000 | 17533822 | 250835730 |
| 17 | 2/41 | 2/41 | 26188742019 | 156526044 | 22850208765 | 36653572 | 95456828 | 25795686 | 285196512 |
| 18 | 10/47 | 10/47 | 128905499538 | 395039762 | 347811830704 | 4801018177 | 37853546276 | 55040708 | 2274754862 |
| 19 | 3/65 | 3/65 | 37240914906 | 323952200 | 60416567929 | 438886020 | 655163798 | 74083150 | 502884800 |
| 20 | 6/11 | 6/11 | 65055257669 | 956868013 | 87019726621 | 280431671 | 2140045358 | 183597888 | 786803165 |
| 21 | 5/20 | 5/20 | 33779967099 | 216500015 | 42914415082 | 167316287 | 6331145727 | 54642451 | 464246324 |
| 22 | 5/08 | 5/08 | 65340414317 | 18651750 | 38180659776 | 110986117 | 264435033 | 77909563 | 888386856 |
| 23 | 4/83 | 4/83 | 42478771612 | 158083195 | 34238312340 | 252575225 | 1003502567 | 54213650 | 569571327 |
| 24 | 8/15 | 8/15 | 71571371577 | 361759561 | 87004388691 | 411113972 | 3736761118 | 103230363 | 1050065978 |
| 25 | 3/76 | 3/76 | 54213272983 | 22444388 | 79434731566 | 667839912 | 812334511 | 30712579 | 647567555 |
| 26 | 6/09 | 6/09 | 50123001956 | 11200000 | 67928353656 | 167835824 | 4088977937 | 31106259 | 732310630 |
| 27 | 6/43 | 6/43 | 39630667933 | 312215620 | 120058495531 | 182516921 | 6803910835 | 71588992 | 503878040 |
| 28 | 1/99 | 1/99 | 16768231414 | 0 | 21014347184 | 247737506 | 307547648 | 10880847 | 192116020 |
| 29 | 2/19 | 2/19 | 15472145929 | 48632539 | 15425158663 | 47012222 | 138806808 | 14189010 | 215973701 |
| 30 | 4/31 | 4/31 | 35006321626 | 347902500 | 77370536965 | 143189368 | 1451181795 | 58469649 | 422506536 |
| 31 | 2/99 | 2/99 | 17781897702 | 0 | 14323751994 | 102117551 | 451279645 | 12874799 | 201180472 |
| 32 | 2/22 | 2/22 | 19345107491 | 236219258 | 13766492606 | 123022964 | 1003173193 | 19006713 | 267680990 |
| 33 | 2/33 | 2/33 | 34179992981 | 19140000 | 22633744058 | 65953223 | 32400507 | 55525368 | 240327459 |
| 34 | 4/76 | 4/76 | 47679008534 | 758411389 | 57869744453 | 93079404 | 355824384 | 111001842 | 539621093 |
| 35 | 2/43 | 2/43 | 18436514014 | 212806000 | 19261077765 | 30172214 | 390618794 | 44564778 | 258694781 |
| 36 | 2/53 | 2/53 | 24448986888 | 0 | 14235876897 | 23640583 | 76322000 | 13939052 | 287351270 |
| 37 | 3/44 | 3/44 | 25444865471 | 75793778 | 31392470715 | 100939145 | 1761559332 | 48668845 | 324327450 |
| 38 | 6/11 | 6/11 | 99475328549 | 1624283910 | 102374313333 | 310491516 | 100239722 | 381190572 | 1151126508 |
| 39 | 2/33 | 2/33 | 22740817944 | 5000000 | 21613603741 | 345544868 | 20000000000 | 17890127 | 303011951 |
| 40 | 3/27 | 3/27 | 45478103310 | 15000000 | 18653120532 | 83816503 | 233162000 | 25633080 | 513373118 |
| 41 | 4/74 | 4/74 | 75675456138 | 1023546816 | 85410611350 | 330734410 | 1717749529 | 97544683 | 951481305 |
| 42 | 2/78 | 2/78 | 26679014980 | 10000000 | 20275988867 | 99326563 | 35381332 | 16247415 | 308080148 |
| 43 | 1/34 | 1/34 | 15429019143 | 120000 | 8481463453 | 113774340 | 60354794 | 4969623 | 204131635 |
| 44 | 2/02 | 2/02 | 16759343970 | 0 | 16563435306 | 31453301 | 2100000 | 13639980 | 50371017 |
| 45 | 17/18 | 17/18 | 135557497455 | 337969932 | 161760513604 | 405355027 | 15111611772 | 91378152 | 1694107282 |

| The number of unit | Primary personnel | Middle personnel | Deposits | Other resorces | Granted facilites | Interest earned | Deferred claims | Fee recived | Profits paid |
|---|---|---|---|---|---|---|---|---|---|
| $DMU_j$ | $x_j$ | $l_j$ | $z_{1j}$ | $z_{2j}$ | $y_{1j}^{D1}$ | $y_{2j}^{D1}$ | $y_j^{U1}$ | $y_j^{D2}$ | $y_j^{U2}$ |
| 46 | 3/02 | 3/02 | 19630698338 | 0 | 21224530821 | 73945351 | 41414059 | 7680933 | 190578258 |
| 47 | 3/06 | 3/06 | 35867880129 | 156848405 | 36565325297 | 159263836 | 1882957 | 49149445 | 431865745 |
| 48 | 2/84 | 2/84 | 24245450096 | 1000000 | 27052155548 | 201425441 | 82385214 | 25889624 | 282195319 |
| 49 | 3/42 | 3/42 | 20036692851 | 5000000 | 30392894598 | 250147649 | 73706784 | 9503699 | 213896431 |
| 50 | 2/38 | 2/38 | 27320131815 | 44400000 | 30782208801 | 157393926 | 156214731 | 6190712 | 237636396 |

**Table (2) The efficiency of units prior to the allocation, by utilizing the proposed model (15)**

| The efficiency of units prior to the allocation | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **The number of DMU** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
| **The total efficiency** | 0.380 | 0.891 | 1.000 | 1.000 | 0.582 | 0.313 | 0.452 | 1.000 | 0.336 | 0.678 |
| **the efficiency of sub-units $S_1$** | 0.210 | 0.183 | 0.344 | 0.355 | 0.137 | 0.193 | 0.166 | 0.291 | 0.325 | 0.422 |
| **the efficiency of sub-units $S_{21}$** | 0.380 | 0.891 | 1.000 | 1.000 | 0.582 | 0.313 | 0.452 | 1.000 | 0.336 | 0.678 |
| **the efficiency of sub-units $S_{22}$** | 0.314 | 0.050 | 0.530 | 0.218 | 0.237 | 0.037 | 0.203 | 0.152 | 0.265 | 0.141 |
| The number of DMU | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| **The total efficiency** | 0.681 | 0.652 | 0.415 | 0.349 | 0.616 | 0.872 | 0.288 | 1.000 | 0.704 | 0.705 |
| **the efficiency of sub-units $S_1$** | 0.184 | 0.264 | 0.152 | 0.268 | 0.305 | 0.320 | 0.429 | 0.379 | 0.461 | 0.617 |
| **the efficiency of sub-units $S_{21}$** | 0.681 | 0.652 | 0.546 | 0.286 | 0.616 | 0.872 | 0.288 | 1.000 | 0.704 | 0.293 |
| **the efficiency of sub-units $S_{22}$** | 0.433 | 0.122 | 0.415 | 0.349 | 0.192 | 0.214 | 0.273 | 0.118 | 0.445 | 0.705 |
| The number of DMU | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| **The total efficiency** | 0.355 | 0.265 | 0.301 | 0.350 | 0.953 | 0.286 | 0.429 | 0.582 | 0.230 | 0.495 |
| **the efficiency of sub-units $S_1$** | 0.262 | 0.352 | 0.305 | 0.329 | 0.399 | 0.224 | 0.268 | 0.225 | 0.236 | 0.388 |
| **the efficiency of sub-units $S_{21}$** | 0.232 | 0.240 | 0.301 | 0.350 | 0.953 | 0.286 | 0.457 | 0.582 | 0.230 | 0.495 |
| **the efficiency of sub-units $S_{22}$** | 0.355 | 0.265 | 0.287 | 0.301 | 0.144 | 0.130 | 0.429 | 0.171 | 0.204 | 0.429 |
| The number of DMU | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| **The total efficiency** | 0.204 | 0.269 | 0.698 | 0.621 | 0.520 | 0.171 | 0.453 | 1.000 | 0.324 | 0.209 |
| **the efficiency of sub-units $S_1$** | 0.159 | 0.459 | 0.410 | 0.606 | 0.389 | 0.258 | 0.244 | 1.000 | 0.266 | 0.382 |
| **the efficiency of sub-units $S_{21}$** | 0.204 | 0.269 | 0.286 | 0.247 | 0.213 | 0.171 | 0.304 | 0.254 | 0.324 | 0.209 |
| **the efficiency of sub-units $S_{22}$** | 0.193 | 0.218 | 0.698 | 0.621 | 0.520 | 0.146 | 0.453 | 1.000 | 0.178 | 0.151 |
| The number of DMU | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| **The total efficiency** | 0.570 | 0.280 | 0.391 | 0.818 | 0.229 | 0.242 | 0.734 | 0.432 | 0.423 | 0.495 |
| **the efficiency of sub-units $S_1$** | 0.875 | 0.264 | 0.309 | 0.222 | 0.072 | 0.174 | 0.422 | 0.229 | 0.160 | 0.346 |
| **the efficiency of sub-units $S_{21}$** | 0.570 | 0.280 | 0.391 | 0.328 | 0.229 | 0.242 | 0.734 | 0.432 | 0.423 | 0.495 |
| **the efficiency of sub-units $S_{22}$** | 0.346 | 0.159 | 0.074 | 0.818 | 0.163 | 0.129 | 0.347 | 0.277 | 0.142 | 0.081 |

In being attentive to the results which can be observed in Table (2), the total efficiency of not one (bank) branch is equivalent to 1. But in units 3, 4, 8. 18 and 38, the total efficiency value is extremely proximate to 1. Whereas, on one hand branch 36, has the minimum total efficiency and on the other hand, branch 3 has the maximum total efficiency value. Branches 5 and 38, have the minimum and maximum efficiency scores, respectively, in stage 1. In the same manner it can be easily determined that, branches 2 and 3 have the optimum and branch 40 is of the minimum and weakest efficiency in the sub-unit 1, of the second stage. Equally, in relative to the other branches, branches 6 and 38, have the weakest and the most powerful efficiency in the second sub-unit of the second stage in turn. As can be noted from the Tables, branch 38, is efficient in both, the first stage (S1) and in the second sub-unit of the second stage (S22), although, the efficiency score of this branch is low, in the first sub-unit of the second stage (S21). After assessing the efficiency of the entire branches and analyzing their conditions in the contemporary situation, the central branch adopts the policy that, the fixed cost of the headquarters, which is stipulated as being equivalent to 100 units; and is symbolically signified by **c**, in model (22) is for the allocation between 50 branches, in a way that, following the allocation, the total efficiency and the efficiency of the components does not deteriorate, but alleviates as much as possible. Thereby, we will deploy model (22) to calculate this for a fair allocation; and the results which come to hand will be demonstrated in Table (3).

**Table (3) The efficiency of units following allocation by manipulating the proposed model (22)**

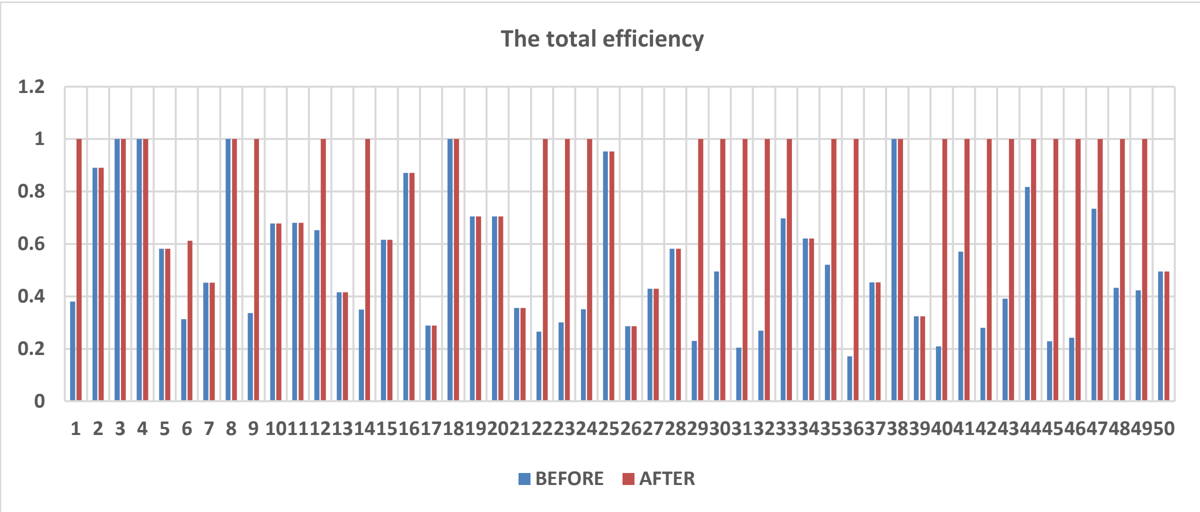
| **The efficiency of units after being allocation** | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **The number of DMU** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
| **The total efficiency** | 1.000 | 0.891 | 1.000 | 1.000 | 0.582 | 0.613 | 0.452 | 1.000 | 1.000 | 0.678 |
| **the efficiency of sub-units $S_1$** | 0.316 | 0.378 | 0.543 | 0.536 | 0.229 | 0.399 | 0.302 | 0.484 | 0.450 | 0.467 |
| **the efficiency of sub-units $S_{21}$** | 0.068 | 0.134 | 0.158 | 0.117 | 0.132 | 0.048 | 0.089 | 0.062 | 0.046 | 0.086 |
| **the efficiency of sub-units $S_{22}$** | 1.000 | 0.891 | 1.000 | 1.000 | 0.582 | 0.613 | 0.452 | 1.000 | 1.000 | 0.678 |
| The number of DMU | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| **The total efficiency** | 0.681 | 1.000 | 0.415 | 1.000 | 0.616 | 0.872 | 0.288 | 1.000 | 0.704 | 0.705 |
| **the efficiency of sub-units $S_1$** | 0.371 | 0.545 | 0.304 | 0.485 | 0.533 | 0.595 | 0.625 | 0.693 | 0.596 | 0.647 |
| **the efficiency of sub-units $S_{21}$** | 0.116 | 0.071 | 0.068 | 0.042 | 0.069 | 0.059 | 0.029 | 0.094 | 0.071 | 0.025 |
| **the efficiency of sub-units $S_{22}$** | 0.681 | 1.000 | 0.415 | 1.000 | 0.616 | 0.872 | 0.288 | 1.000 | 0.704 | 0.705 |
| The number of DMU | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| **The total efficiency** | 0.355 | 1.000 | 1.000 | 1.000 | 0.953 | 0.286 | 0.429 | 0.582 | 1.000 | 1.000 |
| **the efficiency of sub-units $S_1$** | 0.374 | 0.711 | 0.498 | 0.501 | 0.798 | 0.454 | 0.359 | 0.464 | 0.398 | 0.479 |
| **the efficiency of sub-units $S_{21}$** | 0.044 | 0.022 | 0.036 | 0.047 | 0.073 | 0.047 | 0.055 | 0.070 | 0.020 | 0.040 |
| **the efficiency of sub-units $S_{22}$** | 0.355 | 1.000 | 1.000 | 1.000 | 0.953 | 0.286 | 0.429 | 0.582 | 1.000 | 1.000 |
| The number of DMU | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| **The total efficiency** | 1.000 | 1.000 | 1.000 | 0.621 | 1.000 | 1.000 | 0.453 | 1.000 | 0.324 | 1.000 |
| **the efficiency of sub-units $S_1$** | 0.328 | 0.521 | 0.812 | 0.614 | 0.453 | 0.533 | 0.416 | 1.000 | 0.540 | 0.770 |
| **the efficiency of sub-units $S_{21}$** | 0.020 | 0.032 | 0.025 | 0.020 | 0.032 | 0.020 | 0.044 | 0.035 | 0.050 | 0.017 |
| **the efficiency of sub-units $S_{22}$** | 1.000 | 1.000 | 1.000 | 0.621 | 1.000 | 1.000 | 0.453 | 1.000 | 0.324 | 1.000 |
| The number of DMU | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| **The total efficiency** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.495 |
| **the efficiency of sub-units $S_1$** | 0.963 | 0.530 | 0.637 | 0.457 | 0.071 | 0.358 | 0.666 | 0.472 | 0.324 | 0.640 |
| **the efficiency of sub-units $S_{21}$** | 0.041 | 0.031 | 0.033 | 0.034 | 0.164 | 0.040 | 0.040 | 0.052 | 0.070 | 0.047 |
| **the efficiency of sub-units $S_{22}$** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.495 |

As is brought to notice, the results of the total efficiency and the efficiency of the components after being allocated for each of the 50 branches, which have been indicated in Table (3), have enhanced to the utmost possible extent, in comparison, to the efficiency prior to the allocation, though, in some cases, it has remained unmodified.

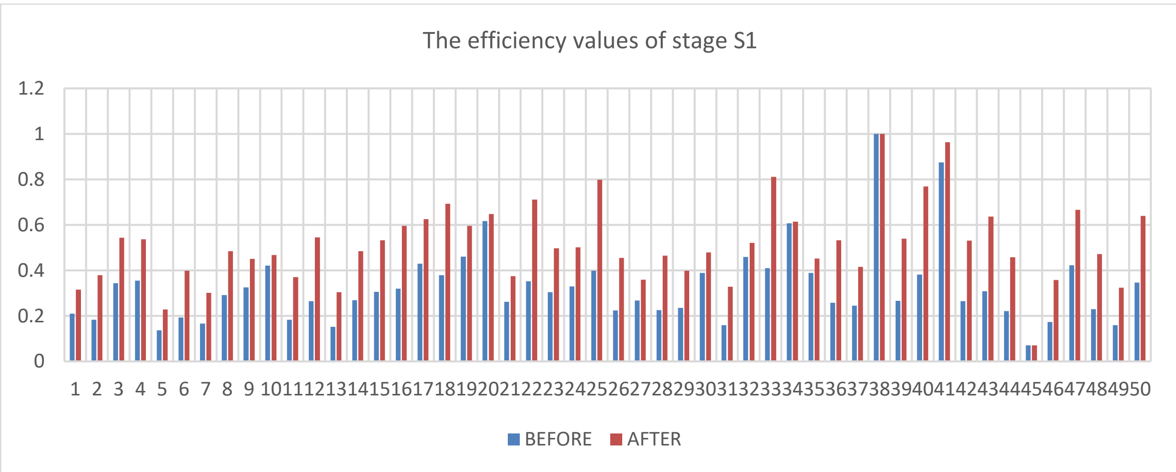
In order to get a better understanding of the efficiency values, which have been achieved from the results obtained, these have been illustrated in the form of a graph, as given below:

The results of Graph (1), shows that, the efficiency of all the units have improved after the allocation; and or some of the units have remained consistent. For instance, in branch (15), the total efficiency prior to the allocation equated to 0.616, whereas, this amount is equivalent to 0.983 after the allocation. It is evident that the efficiency of this branch has improved. Correspondingly, the efficiency of branches 2, 3, 5, 7, 8, 10 and 11 have remained constant after the allocation. Though, in units, such as, 1, 9. 12, 14 and 22, an outstanding increase in efficiency, has caused these units to become virtually efficient succeeding the allocation.

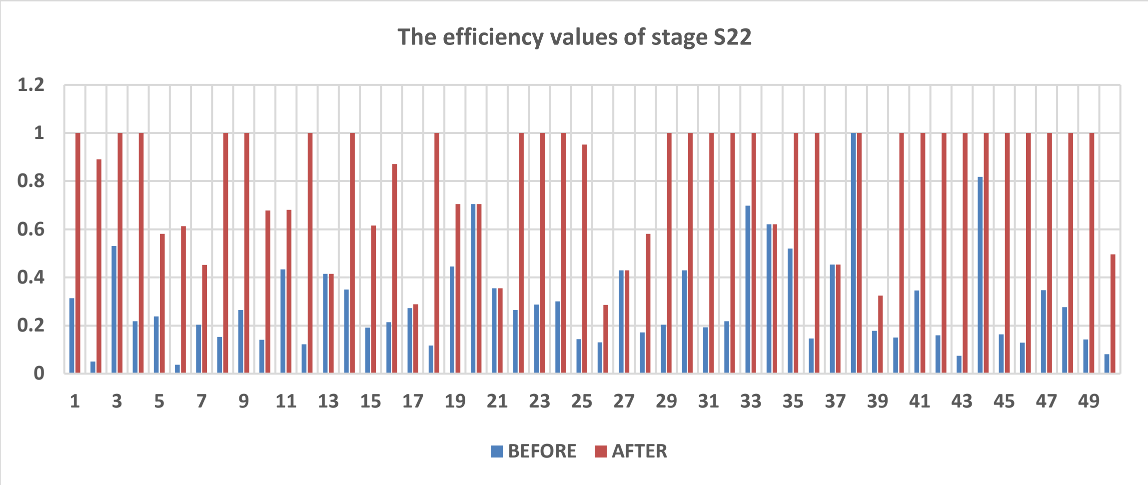
With due attention to Graph (2), the efficiency of the first stage for the entire units have amplified and or remained stable.




**Graph (1) A comparison of the total efficiency values of the branches, which have been attained afore and after the allocation**

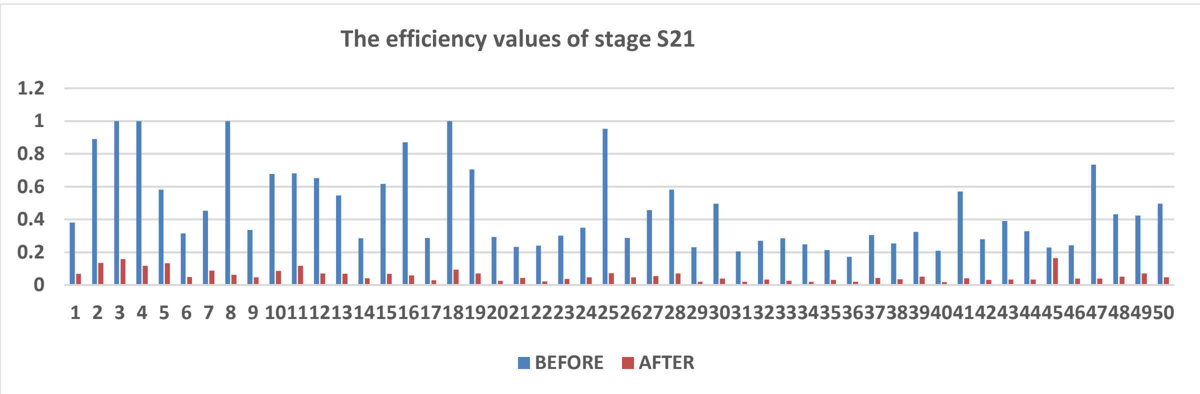



**Graph (2) A comparison of the efficiency values of stage S1 achieved for branches prior to and after the allocation**




**Graph (3) A comparison of the efficiency values of stage S22 that have come to hand for branches before and after the allocation**

As is evident in Fig. (3), the efficiency of the entire units has considerably enhanced and or remained nearly consistent. Likewise, in unit number 2, the efficiency of stages (S1 and S22) before the allocation was 0.183 and 0.053 respectively; whilst, following the allocation, the efficiency score for these two stages equalized to 0.378 and 0.89 for this unit. Hence, it can be observed clearly that, the efficiency of this unit has improved in both the stages. Thus, the allocation has impacted the performance of the total efficiency and the stages (S1 and S22) only. While, the efficiency of stage (S21) in this unit, prior to and after the allocation was 0.89 and 0.134 respectively. This indicates that, the efficiency of stage (S21) in unit 2 has worsened. In comparing Tables (2 and 3), it is perceived that, not only is there no scope for enhancement for any of the units in relative to stage (S21), but that, the efficiency of all the units in this stage have declined. In other words, it can be stated that, the feasibility for improvement in view of this stage, does not exist. By manipulating the available values, in relevance to the total efficiency stipulated for Tables (2 and 3); it can be noted that the performance of branch 22 has experienced no change either after or before the allocation and the total efficiency value has remained steady. In other words, the allocation has not had an impact on the efficiency of this branch or on these two stages.




**Graph (4) A comparison of the efficiency values of stage S21 that have been obtained for branches before and after the allocation**

In Graph (4), it is apparent that following the allocation, the efficiency of none of the units show betterment; and all the units indicate a considerable depreciation in efficiency. Therefore, it can be expressed that in stage (S21), not only is there no opportunity for a progress or enhancement in the efficiency for any of the units, but that the efficiency for the entire units, in this stage, have got much worse. In expressing this in a varied way, there is no probability for alleviation in this stage. This points out to the fact that, we are incapable of conducting the allocation in such a manner, that the efficiency of all the sub-units enhance synchronously and or remain unchanged. So, it seems that a capacity for simultaneous improvement for all the sub-units, in the network structure, with the mentioned characteristics is not available.

## 5. Conclusion

In the assessment of multi-stage units, neglecting the correlations between the sub-units, causes a decrement in the precision of accurate solutions. It is for this purpose and in order to eliminate this issue; as well as being able to deliberate in terms of the fixed cost allocation concept in real DEA applications, this paper investigates the fixed cost allocation in network units. The network which is under study in the current paper, involves an exclusive type, of a two-stage network, where, an additional input and an undesirable output, is apparent within this network. In majority of the studies conducted, self-assessment efficiency scores, are employed, for appraising the efficiency or performance of units, which assesses this efficiency, as being benevolent or optimistic, to a high extent. In the mentioned paper, so as to assess the efficiency of a two-stage network, the common weight approach was utilized, as this technique is a meticulous and acuter survey, when compared to the self-assessment tactic. The model rendered in this paper, presents a fair allocation for two-stage networks, with an additional input and an undesirable output; such that, the total efficiency and the efficiency of the network's sub-units, following the allocation, do not deteriorate as much as possible, but gain improvement, to the utmost degree probable. By taking note of the empirical example, which has been given in section 4, it seems such, that, in the model elucidated, for the assessment of efficiency, in this paper, it is not possible to grant a unique allocation, that does not deteriorate the overall efficiency and the efficiency of all the sub-units synchronously. In view of enhancing the efficiency of the sub-units and in order to unify the optimum solution, in addition to presenting an exclusive and fair allocation, the varied approaches, in relevance with secondary objectives, should be deployed and this issue will be surveyed in papers in the future.

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